The Topological Theory of the Milnor Invariant $\overline{\mu}(1,2,3)$

Lorenzo Leal[1] and Jesús Pineda [2]
*1. Centro de Física Teórica y Computacional,
Facultad de Ciencias, Universidad Central de Venezuela,
AP 47270, Caracas 1041-A, Venezuela.
2. Departamento de Física,
Universidad Simón Bolívar,
Aptdo. 89000, Caracas 1080-A, Venezuela.*

We study a topological Abelian gauge theory that generalizes the Abelian Chern-Simons one, and that leads in a natural way to the Milnor's link invariant $\overline{\mu}(1,2,3)$ when the classical action on-shell is calculated.

I. INTRODUCTION

As it is well known, the vacuum expectation value (v.e.v.) of the Wilson Loop (WL) (or of the product of several WLs) in the Chern-Simons (CS) theory, produces knot (or link) invariants (LIs) [16]. In the Abelian case, the invariants obtained are the Self-Linking Number or the Gauss Linking Number (GLN), depending on whether one deals with one or several Wilson lines. In the non-Abelian case, on the other hand, the invariants obtained are known to be knot or link polynomials, such as the celebrated Jones polynomial [16]. Which polynomial appears depends on the gauge group involved [6].

The GLN of a pair of closed curves admits an analytical expression (see equation 12) that has a simple and appealing geometric interpretation: it represents the oriented number of times that one of the curves flows through a surface bounded by the other one. However, to obtain analytical expressions in the non-Abelian case, one must resort to a perturbative expansion of the v.e.v. of the WLs [5], since as far as it is known, polynomial invariants are not expressible in analytical terms. This perturbative expansion yields an infinite tower of analytical expressions for LIs of increasing complexity. In general, it is not easy to elucidate their geometric or topological meaning, however, the first three of them have been explicitly calculated, and in some cases, a geometrical interpretation is available [2, 5, 6, 7, 9, 12, 15].

A question that naturally rises is about the existence of an intermediate situation between the Abelian and non-Abelian cases. Stated more precisely: is there any topological field theory, other than the Abelian CS (or the Abelian BF theory) that yields *exact* analytical expressions for LIs, other than the GLN?.

Beyond the theoretical interest that this question could have, there is an increasing interest in the description of phenomena that involve closed lines as relevant structures (vortices and defects in condensed matter or fundamental physics, loops in gauge theories and quantum gravity, polymer entanglements, among others examples). Therefore, it could be useful to have at one‘s disposal new topological theories, that while going beyond the Abelian Chern-Simons Theory and its associated GLN, do not present the difficulties of the non-Abelian ones.

The purpose of this paper is to provide an example of such a theory. As we shall see, the LI that the theory we are going to consider produces is the Milnor's Linking Coefficient $\overline{\mu}(1,2,3)$, which is an invariant associated to links of at least three-components. This invariant follows the GLN in an infinite family of link invariants discovered by Milnor several decades ago [13].

The theory that we shall study can be seen as an effective Abelian gauge and diffeomorphism invariant theory, that reproduces just the second contribution of the perturbative expansion of a non-Abelian topological one, namely, of the CS model coupled to chromo-electrically charged particles (so called 'Wong particles' [17]). This Chern-Simons-Wong model has been recently studied from a classical point of view [7, 9].

The action that we are going to deal with comprises a pure gauge-fields part, and terms representing the coupling of these fields with external sources with support on closed curves. The terms that correspond to the gauge-field's part coincide with those of a recent article [4] that studies Chern-Simons theories with non-semisimple group of symmetry. However, in contrast with reference [4], the interaction term that we take is manifestly diffeomorphism invariant, and breaks the non-Abelian gauge invariance of the former term down to an Abelian gauge invariance (see equations (10) and (14)).

In the discussion that follows, we shall adopt the method of dealing with the classical (in the sense of non- quantum-mechanical) theory to calculate LIs [7, 8, 9]. Within this scheme, one solves the equation of motion and calculates the on-shell (OS) action of the topological theory coupled to external Wilson lines. The OS action results to be a functional depending on the Wilson lines that act as sources of the gauge theory and, since the theory is metric independent, it is clearly a LI. For instance, when this procedure is applied to the Abelian CS theory coupled to

Wilson lines, the OS action yields the GLN of the lines, just as in the quantum case. This approach for obtaining LIs from classical field theories can be rigorously proven and generalized to situations where the symmetry group is other than the group of diffeomorphisms of the base manifold [10, 11]. Although we shall focus mainly in this classical approach, we shall also make some remarks about the "quantum method", which is the procedure usually employed to study the relation between LIs and topological theories.

II. THE ACTION AND THE LINK INVARIANT

The action that we shall study is given by

$$\begin{aligned}\mathcal{S} \;=\; & \int d^3x\, \varepsilon^{\mu\nu\rho}\{4\, A^i_\mu(x)\partial_\nu a_{i\rho}(x) + \frac{2}{3}\,\varepsilon^{ijk} a_{i\mu}(x) a_{j\nu}(x) a_{k\rho}(x)\} - 2\int d^3x\; T_i^{\mu x} A^i_\mu(x) + \\ & + \int d^3x \int d^3y\; \varepsilon^{ijk}\, T_i^{\mu x,\nu y} a_{j\mu}(x) a_{k\nu}(y). \end{aligned} \tag{1}$$

Here, $A^i_\mu(x)$ and $a^i_\mu(x)$ are two sets of independent Abelian gauge fields, labelled by Latin letters running from 1 to 3 (we use the summation convention of Einstein also for these "internal" indexes). The first two terms would correspond to the topological theory with non-semisimple gauge group of symmetry introduced in reference [4]. The last two terms in (1) involve the "currents" $T^{\mu x}_{\gamma_i}$ and $T^{\mu x,\nu y}_{\gamma_i}$, with support on the three closed curves γ_i

$$T_i^{\mu y} = \oint_{\gamma_i} dx^\mu \delta^3(x-y), \tag{2}$$

$$T_i^{\mu x,\, \nu y} \equiv \oint_{\gamma_i} dz^\mu \int_0^z dz'^\nu \delta^3(x-z)\delta^3(y-z'). \tag{3}$$

Under general coordinate transformations these objects behave as a vector-density and a bi-local vector density respectively. They obey the differential constraints

$$\partial_\mu T^{\mu y}_{\gamma_i} = 0, \tag{4}$$

$$\frac{\partial}{\partial x^\mu} T^{\mu x,\nu y}_\gamma \;=\; (-\delta^3(x-x_0) + \delta^3(x-y)) T^{\nu y}_\gamma \tag{5}$$

$$\frac{\partial}{\partial y^\nu} T^{\mu x,\nu y}_\gamma \;=\; (\delta^3(y-x_0) - \delta^3(y-x)) T^{\mu x}_\gamma, \tag{6}$$

and the algebraic constraint [2]

$$T^{(\mu x,\nu y)}_\gamma \equiv \frac{1}{2}(T^{\mu x,\nu y}_\gamma + T^{\nu y,\mu x}_\gamma) = T^{\mu x}_\gamma T^{\nu y}_\gamma. \tag{7}$$

(observe that to the action (1) only contributes the antisymmetric part (in μx, νy) of $T^{\mu x,\,\nu y}_{\gamma_a}$).

The "loop coordinates" $T^{\mu y}_\gamma$ and $T^{\mu x,\,\nu y}_\gamma$ are the first members of an infinite sequence that arises when the path ordered exponential that defines the Wilson loop is expanded [2]. As we shall see, the presence of the second "loop-coordinate" $T^{\mu x,\,\nu y}_\gamma$ is just what will lead us to obtain a LI beyond the GLN, which only depends on the first "loop-coordinate" $T^{\mu y}_\gamma$.

Varying the action (1) with respect to A^i_μ and a^i_μ yields

$$\varepsilon^{\mu\nu\rho}\partial_\nu a_{i\rho} = \frac{1}{2} T_i^{\mu x}, \tag{8}$$

$$\varepsilon^{\mu\nu\rho}\partial_\nu A^i_\rho(x) = -\frac{1}{2}\varepsilon^{\mu\nu\rho}\varepsilon^{ijk} a_{j\nu}(x) a_{k\rho}(x) + \frac{1}{2}\int d^3y\; \varepsilon^{ijk} T_j^{[\mu x,\nu y]} a_{k\nu}(y). \tag{9}$$

These equations are just the $0-th$ and first order contributions to the $SU(2)$ Chern-Simons-Wong equations of motion that were studied in references [7, 9]. In that approach, the fields A^i_μ and a^i_μ correspond, respectively, to the first and $0-th$ contributions of a perturbative expansion for the non-Abelian potential [7, 9].

Since $T^{\mu y}_{\gamma_i}$ is divergenceless, equation (8) is consistent. This reflects the invariance of the action under the gauge transformations

$$A^i_\mu \longrightarrow A^i_\mu + \partial_\mu \Lambda^i. \tag{10}$$

The consistency of equation (9) is more involved. Taking the divergence of both sides of this equation yields

$$0 = 2\varepsilon^{\mu\nu\rho}\varepsilon^{ijk}\partial_\mu(a_{j\nu}(x)a_{k\rho}(x)) - \varepsilon^{ijk}\int d^3y\, a_{k\nu}(y)\frac{\partial}{\partial x^\mu}T_j^{[\mu x,\nu y]}. \tag{11}$$

Using the differential constraints (6) and the equation of motion (8), we obtain

$$\varepsilon^{ijk}\delta^3(x - x_j(0))\oint_j dx^\nu \oint_k dz^\beta\, \varepsilon_{\nu\alpha\beta}\frac{(x-z)^\alpha}{|x-z|^3} = \varepsilon^{ijk}\delta^3(x - x_j(0))L(j,k) = 0,$$

were

$$L(i,j) = \frac{1}{4\pi}\oint_{\gamma_i} dz_i^\beta \oint_{\gamma_j} dy_j^\mu\, \varepsilon_{\mu\beta\gamma}\frac{(z-y)^\gamma}{|z-y|^3}, \tag{12}$$

is the GLN between the curves γ_i and γ_j. In the case where these curves do not intersect each other, equation (12) demands that

$$L(i,j) = 0 \quad \forall\; i,j. \tag{13}$$

From this result we obtain that the theory is consistent whenever the curves are not linked in the sense of the GLN. This does not mean that the curves are equivalent to the trivial link (the unlink). For instance, the Borromean Rings are a well known set of three curves whose GLNs vanish, although they are indeed entangled [14]. There are more complex entanglement patterns associated with CS theory than those measured by the GLN.

The consistency condition (13) is also related to a gauge symmetry of the theory. A direct calculation shows that the action (1) is invariant under the transformations

$$a_{i\mu} \to a_{i\mu} + \partial_\mu \Omega_i, \tag{14}$$

provided that the consistency condition (13) is fulfilled. Thus, we see that both sets of fields A_i and a_i must be Abelian gauge fields for the theory to be consistent.

On the other hand, there is no need of introducing a metric in the manifold to construct the action, as can be easily verified. Hence, the theory is metric independent. Since it is also generally covariant, it is a topological theory, just like its cousins the Abelian and non-Abelian Chern-Simons theories. Hence, following references [7, 8, 9], we conclude that the on-shell action S_{os} of the theory should only depend on topological features of the curves appearing in the action, i.e., it should be a link invariant. Let us see how this happens.

The solution of equation (8) is given by

$$a_{i\mu}(x) = -\Big(\frac{1}{2}\Big)\frac{1}{4\pi}\oint_{\gamma_i} dz^\rho\, \varepsilon_{\mu\nu\rho}\frac{(x-z)^\nu}{|x-z|^3}. \tag{15}$$

Equation (9) can also be integrated as easily as the former one, but in order to calculate S_{os} it suffices to substitute the left hand side of (9) and expression (15) into (1). The result is then

$$\begin{aligned} S^{(1)}(1,2,3) \;=\; & -\frac{1}{2}\int d^3x\, \varepsilon^{\mu\nu\rho}a_{1\mu}(x)a_{2\nu}(x)a_{3\rho}(x) - \frac{1}{2}\int d^3x \int d^3y\, \Big(T_1^{[\mu x,\nu y]}a_{2\mu}(x)a_{3\nu}(y) + \\ & + T_2^{[\mu x,\nu y]}a_{3\mu}(x)a_{1\nu}(y) + T_3^{[\mu x,\nu y]}a_{1\mu}(x)a_{2\nu}(y)\Big). \end{aligned} \tag{16}$$

Equation (16) corresponds to an analytical expression for the Milnor's Linking Coefficient $\overline{\mu}(1,2,3)$ [13]. In [9] it was shown by explicit calculation that this expression, despite its appearance, is metric independent, as it should be. An interpretation of its geometrical meaning can be found in references [9, 12].

A sketch of the interpretation of this result would be as follows: the first term in equation (16) measures how many times three arbitrary surfaces whose boundaries are the three curves of the theory (known as Seifert surfaces

[14]) intersect at a common point. The second term counts the oriented number of times that one of the curves crosses first the surface bounded by the second curve and then the surface bounded by the last one. The fact that expression (16) keeps memory of the order in which each curve flows through the surfaces attached to the other curves is what distinguishes this invariant from the GLN. This feature makes $\overline{\mu}(1,2,3)$ a natural "next level" of complexity LI when compared to the GLN. Obviously, further developments along these lines might provide even more (and more interesting) link invariants.

To conclude let us briefly discuss the quantum formulation of the theory, within the Feynman path-integral framework. We consider the functional integral

$$W(\gamma_i) = \int \mathcal{D}A \int \mathcal{D}a \, \exp\left(-S\right). \tag{17}$$

It should be noticed that action (1) already depends on the "Wilson lines" γ_i. This dependence is mandatory to preserve the Abelian gauge-invariance given by (10) and (14). This contrasts with what occurs in the usual Abelian Chern-Simons theory and the non-Abelian one, where gauge invariance does not demand the coupling with external Wilson lines. Integrating out the fields A produces a functional "delta function"

$$\int \mathcal{D}A \, \exp\Big\{ \int d^3x \; A^i_\mu(x) \big\{ 4\,\varepsilon^{\mu\nu\rho}\partial_\nu a_{i\rho}(x) - 2T_i^{\mu x} \big\} \Big\} \propto \; \delta[4\,\varepsilon^{\mu\nu\rho}\partial_\nu a_{i\rho}(x) - 2T_i^{\mu x}], \tag{18}$$

that when substituted into (17) enforces the a fields to take their on-shell values given by (15). Hence, the result is

$$W(\gamma_i) = C \exp(S_{os}), \tag{19}$$

where C is a constant and S_{os} is the link invariant given in equation (16).

In the preceding discussion we have ignored that, indeed, gauge invariance leads to infinities in the Feynman path-integrals that should be properly handled. This can be done, for instance, by employing the Faddeev-Popov method [3] in the usual way. The result still is given by (19), as can be readily checked.

Summarizing, we have presented a topological model that "interpolates" between Abelian and Non-Abelian Chern-Simons theory, in the sense that it leads to a link invariant that goes beyond the GLN yielded by the Abelian theory, but otherwise retaining the property of being an exactly soluble model, unlike the non-Abelian one. The link invariant so obtained corresponds to the Milnor linking coefficient $\overline{\mu}(1,2,3)$.

This work was supported by Proyect PI 03-00-6316-2006 of *Consejo de Desarrollo Científico y Humanístico*, Universidad Central de Venezuela, Caracas, VENEZUELA, and by Project G-2001000712 of FONACIT, VENEZUELA.

[1] A.P. Balachandran, M. Borchardt and A. Stern, Phys. Rev. **D17**, 3247 (1978).
[2] Di Bartolo, C., Gambini, R., Griego, J. *Commun. Math. Phys.* **158**, 217 (1993).
[3] L. D. Faddeev and V. N. Popov, Phys. Lett. B **25**, 29 (1967).
[4] Ferrari, F. *J. Math. Phys.* **44**, 1 138–145 (2003).
[5] Guadagnini, E., Martellini, M., Mintchev, M. *Nucl. Phys.* **B330**, 575 (1990).
[6] J.M.F. Labastida, *Chern-Simons Gauge Theory: Ten Years Later*, hep-th/9905057, USC-FT-7/99, in "*Trends in Theoretical Physics II*", Buenos Aires, Argentina, (1998).
[7] L. Leal, *Knot Invariants from Classical Field Theories*, hep-th/9911133, UCVFC-DF/17-99, in "*Spanish Relativity Meeting*", Bilbao, Spain, (1999).
[8] L. Leal, Mod. Phys. Lett. **A7**, 541 (1992).
[9] L. Leal, Phys. Rev. D. **66**, 125007 (2002).
[10] R. Díaz and L. Leal, in preparation.
[11] R.Díaz, E.Fuenmayor, L.Leal; Phys. Rev. D. **73**, 065012 (2006).
[12] B. Mellor and P. Melvin, *Geometric interpretation of* $\overline{\mu}(ijk)$, math.GT/0110001, (2001).
[13] J. Milnor, Ann. of Math. **59**, 177 (1954).
[14] D. Rolfsen, *Knots and Links*, Wilmington, Publish or Perish (1976).
[15] L. Rozansky, J. Math. Phys. **35**, 5219 (1994).
[16] Witten, E. *Commun. Math. Phys.* **121**, 351–399 (1989).
[17] S. K. Wong, Nuovo Cimento **A65**, 689 (1970).